

Heteroclinic Orbits for a Discrete Pendulum Equation*

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Abstract

About twenty years ago, Rabinowitz showed firstly that there exist heteroclinic orbits of autonomous Hamiltonian system joining two equilibria. A special case of autonomous Hamiltonian system is the classical pendulum equation. The phase plane analysis of pendulum equation shows the existence of heteroclinic orbits joining two equilibria, which coincide with the result of Rabinowitz. However, the phase plane of discrete pendulum equation is similar to that of the classical pendulum equation, which suggests the existence of heteroclinic orbits for discrete pendulum equation also. By using variational method and delicate analysis technique, we show that there indeed exist heteroclinic orbits of discrete pendulum equation joining every two adjacent points of $\{2k\pi + \pi : k \in \mathbb{Z}\}$.

Key Words and Phrases: heteroclinic solution, critical point, discrete pendulum equation, minimization arguments.

1 Introduction

Let us now introduce some notations that will be used throughout this paper. By $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}^+$ we denote the sets of all natural numbers, integers, real numbers and positive real numbers, respectively. For $a, b \in \mathbb{Z}$ ($a \leq b$), define *integer intervals* $Z[a] = \{a, a+1, a+2, \dots\}$, $Z[a, b] = \{a, a+1, \dots, b\}$. For $D \subset \mathbb{R}$, $\varepsilon > 0$, denote by $B_\varepsilon(D)$ the open ε -neighborhood of D . For a convergent bi-infinite sequence $\{x_n\}_{n=-\infty}^{\infty}$, denote by $x_{\pm\infty}$ the limits of the sequence as n tends to $\pm\infty$, i.e. $x_{+\infty} := \lim_{n \rightarrow +\infty} x_n$ and $x_{-\infty} := \lim_{n \rightarrow -\infty} x_n$.

Consider the following second order equation

$$\Delta^2 x_{n-1} + A \sin x_n = 0, \quad n \in \mathbb{Z}, \quad (1)$$

where $A \in \mathbb{R}^+$, $x_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$. A solution $x : \mathbb{Z} \rightarrow \mathbb{R}$ of (1) is called a *heteroclinic solution* (or *heteroclinic orbit*) if there exist $\xi, \eta \in \mathbb{R}$, $\xi \neq \eta$, such that ξ, η are two equilibria of (1) and $x_{-\infty} = \xi$, $x_{+\infty} = \eta$.

We are interested in the problem of the existence and multiplicity heteroclinic solutions of (1). So far as we are aware, it is the first time in the literature for us to study heteroclinic orbits of difference equations.

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Equation (1) can be considered as a discrete analogue of the classical pendulum equation:

$$x''(t) + A \sin x(t) = 0, \quad t \in \mathbb{R}. \quad (2)$$

The phase plane portrait of (1) with $A = 0.1$, shown on Figure ??, can be compared to the phase plane portrait of (2) with $A = 1$, shown on Figure ?. The phase plane analysis of (2) shows the existence of two heteroclinic solutions for (2) joining $-\pi$ and π . On the other hand, the phase plane of (1) is similar to that of (2). On Figure ??, we use colors to distinguish between different orbits. Nine ellipses represent nine periodic orbits, while two curves around nine ellipses are non-periodic orbits. Close similarities observed on Figures ?? and ?? suggest the existence of heteroclinic orbits for (1). Our goal in this paper is to show that there indeed exist two heteroclinic solutions of (1) joining $-\pi$ and π also.

Let us now recall briefly the existence and multiplicity heteroclinic orbits for the following Hamiltonian system

$$q'' + V'_q(t, q) = f(t), \quad (3)$$

which is a generalization form of (2), where $q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$, $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}^n$. In the past twenty years, many authors had studied the existence and multiplicity of heteroclinic solutions and heteroclinic chains for (3). The first result in this area was proved in [14], where the author discussed (3) under assumptions: (V_0) $f(t) = 0$, (V_1) $V \in C^1(\mathbb{R}^n, \mathbb{R})$, and (V_2) V is periodic in q_i with the period T_i , $1 \leq i \leq n$. Conditions (V_1) and (V_2) imply that V has a global maximum on \mathbb{R}^n . Without loss of generality, it is assumed that the global maximum of V is 0 and put $\Gamma = \{\xi \in \mathbb{R}^n : V(\xi) = 0\}$. Under the non-degeneracy condition: (V_3) Γ consists only of isolated points, the following result was obtained in [14].

Theorem A. *Under assumptions $(V_0) - (V_3)$, for every $\xi \in \Gamma$, there exist at least two heteroclinic orbits of (3) joining ξ to $\Gamma \setminus \{\xi\}$. At least one of these orbits emanates from ξ and at least one terminates at ξ .*

Let $n = 1$, $V(t, x) = A \sin x$ and $f(t) = 0$. Then (3) becomes (2), and $\Gamma = \{2k\pi + \pi : k \in \mathbb{Z}\}$. Theorem A guarantees at least two heteroclinic orbits of (2) through every point of Γ .

Further development in this direction was done by Felmer (cf. [7]), who generalized the above result to first order spatially periodic Hamiltonian systems. By using the saddle point theorem, the author obtained the existence of heteroclinic orbits joining two saddle type critical points. Without imposing the non-degeneracy condition, Caldiroli and Jeanjean (cf. [3]) studied conservative singular Hamiltonian system without forcing term, and were able to establish the existence of heteroclinic orbits joining a global maximum point and a non-constant periodic solution.

In the case when the potential is periodic and time reversible, by using minimization arguments, Rabinowitz showed the existence of heteroclinic solutions between pairs of periodic solutions, (cf. [11, 12]). Under the same assumptions, Maxwell (cf. [10]) proved that there exist heteroclinic chains connecting every pairs of periodic solutions.

For non-autonomous Hamiltonian systems without forcing term, Strobel (cf. [16]) studied the existence of heteroclinic chains between pairs of equilibria. By using constrained minimization and comparison arguments, Rabinowitz and Zelati (cf. [15]) studied (3) without forcing term, and found multiple heteroclinic chains joining pairs of equilibria. Subsequent progress was done by Bertotti and Montecchiari (cf. [2]), who generalized the results of Strobel, by proving the existence of infinitely many heteroclinic solutions for a class of forced slowly oscillating Hamiltonian system with potential $V(t, x)$ of form $\alpha(t)W(x)$, with α being almost periodic in t . Next, Alessio, Bertotti

and Montecchiari (cf. [1]) obtained a generalization of the results of [2], in which $\alpha(t)$ is replaced by $\alpha(t) + \alpha(\varepsilon t)$ for $\varepsilon > 0$ small enough. However, without the non-degeneracy, these results are not as strong as those of Strobel. In the case of forced slowly oscillating Hamiltonian systems, Rabinowitz (cf. [13]) showed the existence of basic and even more complex heteroclinic orbits, without making any non-degeneracy assumption. Then, Zelati and Rabinowitz (cf. [17]) showed that there exist heteroclinic solutions joining two stationary points in different energy levels, which was established by using minimization arguments.

We should also mention the work by Chen and Tzend (cf. [4, 5, 6]), in which variational and penalization methods were being used to study the existence of heteroclinic orbits for the following system

$$q'' - V_q(t, q) = 0, \quad (4)$$

where V is not periodic nor asymptotically periodic in t . In those papers, the authors obtained multiple heteroclinic orbits and chains joining pairs of equilibria as well as joining an equilibrium to a non-constant periodic solution. Izydorek and Janczewska (cf. [9]) proved, without assuming periodicity or almost periodicity in t of the potential, for (3) without forcing term, the existence of heteroclinic solutions joining pairs of equilibria.

However, no results on the existence of heteroclinic solutions of difference equations have been proved. In this paper, by using variational arguments, we will study the existence and multiplicity of heteroclinic solutions for (1). To this end, we need to choose a suitable functional space on which a variational functional, associated with (1), can be constructed. However, lack of continuity assumption leads to some new problems which were not present in the case of differential systems. For example, for differential systems, if an orbit contains two points such that one of them is outside of $B_\varepsilon(\xi)$, while the other belongs to inside of $B_\delta(\xi)$ ($\delta < \varepsilon/2$), then the orbit (because of its continuity) contains a point belonging to $\partial B_{\varepsilon/2}(\xi)$. However, such a statement is not valid for orbits of discrete systems.

2 Main Results

In this section, we study the existence and multiplicity of heteroclinic orbits of (1) joining every two adjacent points of $\{2k\pi + \pi : k \in \mathbb{Z}\}$. For simplicity, we make an image translation. By applying the substitution $y_n = x_n - \pi$, (1) can be rewritten as

$$\Delta^2 y_{n-1} - A \sin y_n = 0, \quad n \in \mathbb{Z}. \quad (5)$$

We look for heteroclinic orbits of (5) which join two adjacent points of $\{2k\pi : k \in \mathbb{Z}\}$.

Let C be the vector space of all convergent sequences $y = \{y_k\}_{k=-\infty}^{\infty}$, i.e.

$$C := \left\{ y = \{y_k\} : \lim_{k \rightarrow \infty} y_k \text{ and } \lim_{k \rightarrow -\infty} y_k \text{ exist, } y_k \in \mathbb{R}, k \in \mathbb{Z} \right\}.$$

We define the space H by

$$H := \left\{ y \in C : \sum_{k=-\infty}^{\infty} |\Delta y_k|^2 < \infty \right\},$$

and put

$$\langle x, y \rangle := \sum_{k=-\infty}^{\infty} \Delta x_k \Delta y_k + x_0 y_0, \quad \forall x, y \in H, \quad (6)$$

$$\|y\| := \left[\sum_{k=-\infty}^{\infty} (\Delta y_k)^2 + y_0^2 \right]^{\frac{1}{2}}, \quad \forall y \in H. \quad (7)$$

Then we have

Proposition 2.1. *The bilinear product (6) is an inner product on H and the space H equipped with the norm given by (7) is a Hilbert space.*

Proof. Recall that the space $l^2(\mathbb{Z})$ of all sequences $a = \{a_k\}_{k=-\infty}^{\infty}$ such that

$$\|a\|_2 := \left[\sum_{k=-\infty}^{\infty} a_k^2 \right]^{\frac{1}{2}} < \infty,$$

is a Hilbert space. Let $\{y^n\} \subset H$ be a Cauchy sequence in H , i.e.

$$\forall \varepsilon > 0 \exists_N \forall_{m, n \geq N} \|y^n - y^m\| = \left[\sum_{k=-\infty}^{\infty} (\Delta y_k^n - \Delta y_k^m)^2 + (y_0^n - y_0^m)^2 \right]^{\frac{1}{2}} < \varepsilon. \quad (8)$$

Then $\{y_0^n\}$ is a Cauchy sequence in \mathbb{R} , while $\{\Delta y^n\}$, $\Delta y^n := \{\Delta y_k^n\}$, is a Cauchy sequence in $l^2(\mathbb{Z})$. By completeness of $l^2(\mathbb{Z})$, there exists a limit a in $l^2(\mathbb{Z})$ of $\{\Delta y^n\}$. One can easily observe, that there exists a unique $y^0 := \{y_k^0\}$ in H such that

$$\lim_{n \rightarrow \infty} y_0^n = y_0^0, \quad \text{and} \quad \forall_{k \in \mathbb{Z}} \Delta y_k^0 = a_k.$$

By passing to the limit as m goes to ∞ , we obtain from (8)

$$\forall \varepsilon > 0 \exists_N \forall_{n \geq N} \|y^n - y^0\| \leq \varepsilon,$$

which proves that $\{y^n\}$ converges to y^0 . Consequently, H is a Hilbert space. \square

Similar arguments as those presented in [8], one can define variational functional $J : H \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ associated with (5) by

$$J(y) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} |\Delta y_n|^2 + A(1 - \cos y_n) \right]. \quad (9)$$

Put $\Theta := \{2k\pi : k \in \mathbb{Z}\}$ and $\gamma := \frac{2\pi}{3}$.

Remark 2.1. For every $y = \{y_n\} \in H$, if $J(y) < \infty$, then $y_{-\infty}, y_{+\infty} \in \Theta$. Indeed, suppose for example $y_{+\infty} \notin \Theta$, then there exist $\gamma > \delta > 0$ and $N \in \mathbb{N}$ such that $y_n \notin B_\delta(\Theta)$ for all $n \geq N$. Therefore,

$$J(y) \geq \sum_{i=N}^{\infty} A(1 - \cos y_i) \geq \sum_{i=N}^{\infty} A(1 - \cos \delta) = \infty.$$

Given $\xi \in \Theta \setminus \{0\}$, $\varepsilon \in (0, \gamma)$, define the set $\Gamma_\varepsilon(\xi)$ of all $y \in H$ satisfying

- (i) $y_{-\infty} = 0$,
- (ii) $y_{+\infty} = \xi$,
- (iii) $y_n \notin B_\varepsilon(\Theta \setminus \{0, \xi\})$ for all $n \in \mathbb{Z}$.

Obviously, $\Gamma_\varepsilon(\xi)$ is not empty for all $\xi \in \Theta$. Define

$$c_\varepsilon(\xi) := \inf_{y \in \Gamma_\varepsilon(\xi)} J(y) \quad \text{and} \quad \alpha_\varepsilon := \min_{t \notin B_\varepsilon(\Theta)} (1 - \cos t) > 0.$$

Now we give a simple but useful lemma.

Lemma 2.1. *Given a sequence of disjoint integer intervals $Z(n_k, m_k)$, $n_k < m_k$ and $j \in \mathbb{N}$. Let $y \in H$ be such that*

$$y_i \notin B_\varepsilon(\Theta) \quad \text{for} \quad i \in \bigcup_{k=1}^j Z(n_k, m_k).$$

Then,

$$J(y) \geq \sqrt{2A\alpha_\varepsilon} \sum_{k=1}^j |y_{m_k} - y_{n_k}|.$$

Proof. Let $l = \sum_{k=1}^j |y_{m_k} - y_{n_k}|$. Since for $m_k \geq n_k + 1$

$$\left| \sum_{i=n_k}^{m_k-1} \Delta y_i \right| \leq \sum_{i=n_k}^{m_k-1} |\Delta y_i| \leq \sqrt{m_k - n_k - 1} \left(\sum_{i=n_k}^{m_k-1} |\Delta y_i|^2 \right)^{\frac{1}{2}}, \quad (10)$$

and since $(1 - \cos y_i) \geq 0$ and $y_i \notin B_\varepsilon(\Theta)$ for $i \in Z(n_k, m_k)$, we have

$$\begin{aligned} J(y) &\geq \frac{1}{2} \sum_{k=1}^j \sum_{i=n_k}^{m_k-1} |\Delta y_i|^2 + \sum_{k=1}^j \sum_{i=n_k}^{m_k-1} A(1 - \cos y_i) \\ &\geq \sum_{k=1}^j \left(\frac{|y_{m_k} - y_{n_k}|^2}{2r_k} + A\alpha_\varepsilon r_k \right) \\ &\geq \sum_{k=1}^j \sqrt{2A\alpha_\varepsilon} |y_{m_k} - y_{n_k}|, \end{aligned}$$

$$\text{where } r_k := \begin{cases} m_k - n_k - 1 & \text{if } m_k > n_k + 1 \\ 1 & \text{if } m_k = n_k + 1 \end{cases}. \quad \square$$

Assume $0 < \varepsilon < \gamma$. We will prove the existence of an orbit minimizing function J restricted to $\Gamma_\varepsilon(\xi)$. For this purpose, we need the following two lemmas.

Lemma 2.2. *Consider $\xi \in \Theta \setminus \{0\}$ and assume that $\{y^m\}_{m=1}^\infty \subset H$ is a minimizing sequence for (9) restricted to $\Gamma_\varepsilon(\xi)$, such that for any $n \in \mathbb{N}$, $y^m \rightarrow y$ uniformly for $i \in Z[-n, n]$. If $y \in H$ and $J(y) < \infty$, then $y \in \Gamma_\varepsilon(\xi)$.*

Proof. By Remark 2.1, there exist $\zeta, \eta \in \Theta$ such that $y_{-\infty} = \zeta$, $y_{+\infty} = \eta$. By assumption $y^m \rightarrow y$ uniformly for $i \in Z[-n, n]$ and $y^m \in \Gamma_\varepsilon(\xi)$.

Claim 1: $y_n \notin B_\varepsilon(\Theta \setminus \{0, \xi\})$ for all $n \in \mathbb{N}$.

Indeed, if there exist $n_0 \in \mathbb{N}$ and $\theta \in \Theta \setminus \{0, \xi\}$ such that $y_{n_0} \in B_\varepsilon(\theta)$, then $\delta := |y_{n_0} - \theta| < \varepsilon$. Since $y^m \rightarrow y$ uniformly for $i \in Z[-n_0, n_0]$, we have for sufficiently large m that $|y_{n_0}^m - y_{n_0}| < \varepsilon - \delta$ and $|y_{n_0}^m - \theta| \leq |y_{n_0}^m - y_{n_0}| + |\theta - y_{n_0}| < \varepsilon$, which is a contradiction.

Claim 2: $y_{\pm\infty} \in \{0, \xi\}$.

If $y_{-\infty} = \zeta \in \Theta \setminus \{0, \xi\}$, then for any $0 < \varepsilon_1 \leq \varepsilon$, $\exists N_1 \in \mathbb{N} \forall n \geq N_1$ $y_{-n} \in B_{\varepsilon_1/2}(\zeta)$. Since $y^m \rightarrow y$ uniformly for $n \in Z[-N_1, N_1]$, there exists $N_2 \in \mathbb{N}$, $|y_{N_1}^m - y_{-N_1}| < \varepsilon_1/2$ for $\forall m > N_2$. Consequently, for those ε_1, N_1, N_2 and $m > N_2$, we have $|y_{N_1}^m - \zeta| \leq |y_{N_1}^m - y_{-N_1}| + |y_{-N_1} - \zeta| < \varepsilon_1 \leq \varepsilon$. Thus $y_{N_1}^m \in B_\varepsilon(\zeta)$, which contradicts the fact that $y^m \in \Gamma_\varepsilon(\xi)$. Thus $\zeta \in \{0, \xi\}$. A similar argument can be applied to show $\eta \in \{0, \xi\}$.

Claim 3: $y_{-\infty} = 0$.

Since $y^m \in \Gamma_\varepsilon(\xi)$, for every $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{Z}$ such that $y_{n(m)+1}^m \notin B_\varepsilon(0)$ and $y_n^m \in B_\varepsilon(0)$ for all $n \leq n(m)$. For $y \in H$, put $x_n(m) := y_{n-m}$ and $x(m) = \{x_n(m)\}$. Then $J(x(m)) = J(y)$. Therefore, we can assume that $n(m) = 0$ for all $m \in \mathbb{N}$. Consequently $y_n^m \in B_\varepsilon(0)$ and $y_n \in \overline{B}_\varepsilon(0)$, $\forall n \leq 0$. Thus, $\zeta \in \overline{B}_\varepsilon(0) \cap \{0, \xi\} = \{0\}$, i.e. $\zeta = 0$.

Claim 4: $y_{+\infty} = \xi$.

Notice that $y_{+\infty} \in \{0, \xi\}$. Choose $\delta > 0$ satisfying $6\delta < \varepsilon$ and $\frac{1}{2}(2\delta)^2 + \delta^2 < \sqrt{2A\alpha_x}\varepsilon/6$. In order to show that such δ exists, put $f(x) = \sqrt{2A\alpha_x}\varepsilon/6 - \frac{1}{2}(2x)^2 - x^2 = \varepsilon\sqrt{A}\sin\frac{x}{2}/3 - 3x^2$. Then, $f'(x) = \varepsilon\sqrt{A}\cos\frac{x}{2}/6 - 6x$, $f(0) = 0$ and there exists $x_0 \in (0, \pi/2)$ such that $f(x) > 0$ for $0 < x < x_0$, which implies the existence of δ with the required properties. Suppose, to the contrary, that $y_{+\infty} = 0$, then there exists $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ $y_{n_0} \notin B_\delta(0)$ and $y_n \in B_\delta(0)$. Since $y^m \rightarrow y$ uniformly for $i \in Z[-n_0 - 1, n_0 + 1]$, there exists a sufficiently large m , such that $|y_{n_0+1}^m - y_{n_0+1}| < \delta$. Thus $y_{n_0+1}^m \in B_{2\delta}(0)$. We need to consider the following two cases:

Case 1: $y_{n_0}^m \notin B_{5\delta}(0)$.

Then $|y_{n_0+1}^m - y_{n_0}^m| > 3\delta$, and we have

$$J(y^m) \geq 9\delta^2/2 + \sum_{n=n_0+1}^{\infty} \left[\frac{1}{2}|\Delta y_n^m|^2 + A(1 - \cos y_n^m) \right].$$

Define

$$x_n^m := \begin{cases} 0, & n \leq n_0 \\ y_n^m, & n \geq n_0 + 1 \end{cases}$$

Then $x^m = \{x_n^m\} \in \Gamma_\varepsilon(\xi)$ and

$$\begin{aligned} J(x^m) &= \sum_{n=n_0}^{\infty} \left[\frac{1}{2}|\Delta x_n^m|^2 + A(1 - \cos x_n^m) \right] \\ &= \frac{1}{2}|y_{n_0+1}^m|^2 + \sum_{n=n_0+1}^{\infty} \left[\frac{1}{2}|\Delta y_n^m|^2 + A(1 - \cos y_n^m) \right] \\ &\leq \frac{1}{2}|y_{n_0+1}^m|^2 + J(y^m) - \frac{9\delta^2}{2} \\ &< J(y^m) - \frac{5\delta^2}{2}, \end{aligned}$$

which leads to the following contradiction

$$c_\varepsilon(\xi) = \lim_{m \rightarrow \infty} J(y^m) \geq \lim_{m \rightarrow \infty} J(x^m) + 5\delta^2/2 \geq c_\varepsilon(\xi) + 5\delta^2/2.$$

Case 2: $y_{n_0}^m \in B_{5\delta}(0)$.

Subcase I: $y_n^m \notin B_\delta(0)$ for all $1 \leq n \leq n_0$.

Then

$$J(y^m) \geq \sqrt{2A\alpha_\delta\varepsilon}/6 + \sum_{n=n_0+1}^{\infty} \left[\frac{1}{2} |\Delta y_n^m|^2 + A(1 - \cos y_n^m) \right]$$

Define

$$z_n^m := \begin{cases} 0, & n \leq n_0 \\ y_n^m, & n \geq n_0 + 1 \end{cases}$$

Then $z^m = \{z_n^m\} \in \Gamma_\varepsilon(\xi)$ and

$$\begin{aligned} J(z^m) &= \sum_{n=n_0}^{\infty} \left[\frac{1}{2} |\Delta z_n^m|^2 + A(1 - \cos z_n^m) \right] \\ &= \frac{1}{2} |y_{n_0+1}^m|^2 + \sum_{n=n_0+1}^{\infty} \left[\frac{1}{2} |\Delta y_n^m|^2 + A(1 - \cos y_n^m) \right] \\ &\leq \frac{1}{2} |y_{n_0+1}^m|^2 + J(y^m) - \sqrt{2A\alpha_\delta\varepsilon}/6 \\ &< J(y^m) - \delta^2, \end{aligned}$$

which yields the following contradiction

$$c_\varepsilon(\xi) = \lim_{m \rightarrow \infty} J(y^m) \geq \lim_{m \rightarrow \infty} J(x^m) + \delta^2 \geq c_\varepsilon(\xi) + \delta^2.$$

Subcase II: There exists a $n_1 \in Z[1, n_0]$ such that $y_{n_1}^m \in B_\delta(0)$, $y_n^m \notin B_\delta(0)$, $\forall n \in Z[1, n_1 - 1]$. If $y_{n_1}^m \notin B_{5\delta}(0)$, by a similar argument as in Case 1, we get a contradiction. On the other hand, if $y_{n_1}^m \in B_{5\delta}(0)$, then by the argument used in Subcase I of Case 2, we again obtain a contradiction.

Consequently, $y \in \Gamma_\varepsilon(\xi)$, which completes the proof. \square

Lemma 2.3. For any $\varepsilon \in (0, \gamma)$, $\xi \in \Theta \setminus \{0\}$, there exists $y^0 := y(\varepsilon, \xi) \in \Gamma_\varepsilon(\xi)$ such that $J(y(\varepsilon, \xi)) = c_\varepsilon(\xi)$, i.e. $y(\varepsilon, \xi)$ minimizes $J|_{\Gamma_\varepsilon(\xi)}$.

Proof. Let $\{y^m\}_{m=1}^\infty$ be a minimizing sequence for (9). There exists a positive number $M > 0$ such that $M \geq J(y^m) \geq \frac{1}{2} \sum_{n=-\infty}^\infty |\Delta y_n^m|^2$. We claim that $\{y_0^m\}_{m=1}^\infty$ is a bounded sequence. Suppose to the contrary that for any $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that $|y_0^{m_k}| \geq k$. Thus $\lim_{k \rightarrow \infty} |y_0^{m_k}| = \infty$, and there exists $k_0 \in \mathbb{N}$ such that $y_0^{m_k} \notin B_\varepsilon(\xi)$ when $k \geq k_0$. Consider $y_1^{m_k}$.

Case I: If $y_1^{m_k} \in \overline{B}_\varepsilon(\xi)$, then $J(y^{m_k}) \geq |y_0^{m_k} - \xi - \varepsilon|^2/2$. Let $k \rightarrow \infty$, we have $J(y^{m_k}) \rightarrow \infty$, which contradicts the assumptions.

Case II: Otherwise, $y_1^{m_k} \notin \overline{B_\varepsilon}(\xi)$. Denote $n_k := \{n > 0 : y_{n+1}^{m_k} \in B_\varepsilon(\xi), y_l^{m_k} \notin B_\varepsilon(\xi), \forall l \in Z[0, n]\}$. Then we have

$$J(y^{m_k}) \geq \sqrt{2A\alpha_\varepsilon}|y_0^{m_k} - y_{n_k}^{m_k}| + \frac{1}{2}|y_{n_k+1}^{m_k} - y_{n_k}^{m_k}|^2 \text{ for all } k > k_0. \quad (11)$$

Let $k \rightarrow \infty$ in (11), then $|y_0^{m_k} - y_{n_k+1}^{m_k}| \rightarrow \infty$. But $|y_0^{m_k} - y_{n_k+1}^{m_k}| \rightarrow \infty$ if and only if $|y_0^{m_k} - y_{n_k}^{m_k}| + |y_0^{m_k} - y_{n_k+1}^{m_k}| \rightarrow \infty$ which is equivalent to $\sqrt{2A\alpha_\varepsilon}|y_0^{m_k} - y_{n_k}^{m_k}| + \frac{1}{2}|y_{n_k+1}^{m_k} - y_{n_k}^{m_k}|^2 \rightarrow \infty$, which contradicts again the assumptions.

Consequently, $\{y_0^{m_k}\}$ is a bounded sequence and, by the definition of the norm on H , $\{y^{m_k}\}$ is a bounded sequence in H . Therefore, passing to a subsequence if necessary, there is $y^0 \in H$ such that y^m weakly converges to y^0 in H .

We claim $J(y^0) < \infty$. Indeed, consider $-\infty < s < t < \infty$ and define for $y \in H$

$$J(s, t, y) = \sum_{n=s}^t \left[\frac{1}{2}|\Delta y_n|^2 + A(1 - \cos y_n) \right].$$

The weak convergence of the sequence $\{y^m\}$ to y^0 in the Hilbert space H implies that $y_n^m \rightarrow y_n^0$ for any $n \in \mathbb{Z}$. Then, $\{y_n^m\}_{n=s}^t$ converges uniformly to $\{y_n^0\}_{n=s}^t$. Clearly, $J(s, t, y)$ is lower continuous, so it is also lower semi-continuous. Combining $M \geq J(y^m) \geq J(s, t, y^m)$ with the lower semi-continuity of $J(s, t, y)$, we have

$$J(s, t, y^0) \leq \liminf_{m \rightarrow \infty} J(s, t, y^m) \leq c_\varepsilon(\xi) = \liminf_{m \rightarrow \infty} J(y^m) \leq M. \quad (12)$$

Since $y^0 \in H$ and s, t are arbitrary, (12) implies $J(y^0) \leq \inf_{y \in \Gamma_\varepsilon(\xi)} J(y)$. Lemma 2.2 implies $y^0 \in \Gamma_\varepsilon(\xi)$, and we have $J(y^0) = c_\varepsilon(\xi)$. \square

Put

$$c_\varepsilon := \inf_{\xi \in \Theta \setminus \{0\}} c_\varepsilon(\xi). \quad (13)$$

We will show that, fixed $\varepsilon > 0$, there are finite $\zeta(\varepsilon)$'s such that $\zeta(\varepsilon) \in \Theta \setminus \{0\}$, $c_\varepsilon(\zeta(\varepsilon))$ is a critical value of J restricted on the set $\bigcup_{\xi \in \Theta} \Gamma_\varepsilon(\xi)$.

Lemma 2.4. *The set $\Upsilon_\varepsilon := \{\xi \in \Theta \setminus \{0\} : c_\varepsilon(\xi) = c_\varepsilon\}$ is finite.*

Proof. Consider $\xi \in \Theta \setminus \{0\}$ and $y \in \Gamma_\varepsilon(\xi)$. Then $y_{-\infty} = 0$, $y_{+\infty} = \xi$, $y_n \notin B_\varepsilon(\Theta \setminus \{0, \xi\})$. Put $m_1 := \max\{n : y_n \in B_\varepsilon(0), y_m \notin B_\varepsilon(0), \forall m > n\}$ and $m_2 = \min\{n : y_n \in B_\varepsilon(\xi), n \geq m_1\}$. If $m_2 > m_1 + 2$, then by Lemma 2.1,

$$J(y) \geq \sum_{n=m_1}^{m_2-1} \frac{1}{2}|\Delta y_n|^2 \geq \sqrt{2A\alpha_\varepsilon}|y_{m_2-1} - y_{m_1+1}| + \frac{1}{2}|\Delta y_{m_1}|^2 + \frac{1}{2}|\Delta y_{m_2-1}|^2.$$

Notice that $\xi \rightarrow \infty$ if and only if $|y_{m_2-1} - y_{m_1+1}| + |\Delta y_{m_1}| + |\Delta y_{m_2-1}| \rightarrow \infty$ which is equivalent to $\sqrt{2A\alpha_\varepsilon}|y_{m_2-1} - y_{m_1+1}| + \frac{1}{2}|\Delta y_{m_1}|^2 + \frac{1}{2}|\Delta y_{m_2-1}|^2 \rightarrow \infty$. Thus $J(y) \rightarrow \infty$ as $\xi \rightarrow \infty$. In the case $m_1 \leq m_2 \leq m_1 + 2$, by a similar (but even simpler) argument, we obtain the same result. Consider $\xi_0 \in \Theta \setminus \{0\}$. Then we have $c_\varepsilon(\xi_0) \geq c_\varepsilon$ and there exists $M_1 > 0$ such that $\inf_{y \in \Gamma_\varepsilon(\xi)} J(y) > c_\varepsilon(\xi_0)$ for all $|\xi| > M_1$. Consequently, there are only finitely many $c_\varepsilon(\xi)$ which can be equal to c_ε . \square

Fixed $\varepsilon > 0$, Lemma 2.4 implies c_ε is achieved at some points $\zeta(\varepsilon) \in \Upsilon_\varepsilon$. Now by choosing a sequence of $\varepsilon_k \rightarrow 0$, we claim that there exists a subsequence $\{\varepsilon_j\}_{j=1}^\infty$ such that, for sufficiently large j the points $\zeta(\varepsilon_j) \in \Upsilon_{\varepsilon_j}$ are independent of j , i.e. we have the following:

Lemma 2.5. *Suppose that ε_k is a decreasing sequence of positive numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a subsequence $\{\varepsilon_j\}_{j=1}^\infty$ such that, for sufficiently large j the points $\zeta(\varepsilon_j) \in \Upsilon_{\varepsilon_j}$ are independent of j .*

Proof. Consider $\{c(\varepsilon_k)\}_{k=1}^\infty$. For any $y \in \Gamma_{\varepsilon_k}(\eta)$, we have $y_n \notin B_{\varepsilon_k}(\Theta \setminus \{0, \eta\})$ and also $y_n \notin B_{\varepsilon_{k+1}}(\Theta \setminus \{0, \eta\})$ for all $n \in \mathbb{Z}$. Thus $y \in \Gamma_{\varepsilon_{k+1}}(\eta)$ and consequently $\Gamma_{\varepsilon_1}(\eta) \subset \Gamma_{\varepsilon_2}(\eta) \subset \dots \subset \Gamma_{\varepsilon_k}(\eta) \subset \Gamma_{\varepsilon_{k+1}}(\eta) \subset \dots$. By definition of $c_{\varepsilon_k}(\eta)$, we have

$$c_{\varepsilon_k}(\eta) = \inf_{y \in \Gamma_{\varepsilon_k}(\eta)} J(y) \geq \inf_{y \in \Gamma_{\varepsilon_{k+1}}(\eta)} J(y) = c_{\varepsilon_{k+1}}(\eta). \quad (14)$$

Thus $\{c_{\varepsilon_k}\}_{k=1}^\infty$ is monotone non-increasing bounded sequence. By a similar argument to the one used in the proof of Lemma 2.4, the sequence $\{\zeta(\varepsilon_k)\}_{k=1}^\infty$ is bounded. Consequently, it contains a convergent subsequence $\{\zeta(\varepsilon_j)\}_{j=1}^\infty$. Since the set Θ consists of isolated points, $\zeta(\varepsilon_j)$ is a constant sequence for j sufficiently large. \square

Since for sufficiently large j the points $\zeta(\varepsilon_j)$ are independent of j , denote by $\zeta = \zeta(\varepsilon_j)$. By Lemma 2.3, there exists $y(\varepsilon_j, \zeta) \in \Gamma_{\varepsilon_j}(\zeta)$ such that $c_{\varepsilon_j} = J(y(\varepsilon_j, \zeta))$.

Theorem 2.1. *For j sufficiently large, $y(\varepsilon_j, \zeta)$ is a heteroclinic solution joining 0 and ζ .*

Proof. Put $y(j) := y(\varepsilon_j, \zeta)$. By the definition of $\Gamma_\varepsilon(\zeta)$ and H , it is sufficient to show that for large j , $y_n(j) \notin \partial B_{\varepsilon_j}(\Theta \setminus \{0, \zeta\})$ for all $n \in \mathbb{Z}$. If not, there would exist a sequence $\eta_k \in \Theta \setminus \{0, \zeta\}$ and $n_k \in \mathbb{Z}$ such that

$$y_{n_k}(k) \in \partial B_{\varepsilon_k}(\eta_k) \text{ and } y_n(k) \notin \partial B_{\varepsilon_k}(\eta_k), \quad \forall n < n_k.$$

By similar argument used in the proof of Lemma 2.4, $\{\eta_k\}$ is bounded. Passing to a subsequence, if necessary, η_k must be a constant sequence, i.e. $\eta_k =: \eta$. We have the following two possibilities:

Case 1: There is an increasing sequence of integers $k' \rightarrow \infty$ such that $y_n(k') \notin \overline{B}_{\varepsilon_j}(\zeta)$ for all $n < n_{k'}$, or

Case 2: For every $j \in \mathbb{N}$ there is a $m_k < n_k$ such that $y_{m_k}(k) \in \partial B_{\varepsilon_k}(\zeta)$.

If Case 1 occurs, define

$$x_n(k') = \begin{cases} y_n(k'), & n \leq n'_k \\ \eta, & n \geq n'_k + 1 \end{cases}$$

Then $y(k') \in \Gamma_{\varepsilon_j}(\eta)$ and

$$\begin{aligned} J(y(k')) - J(x(k')) &= \sum_{n=n_{k'}}^{\infty} \left[\frac{1}{2} |\Delta y_n(k')|^2 + A(1 - \cos y_n(k')) \right] \\ &\quad - \frac{1}{2} |\Delta x_{n_{k'}}(k')|^2 + A(1 - \cos x_{n_{k'}}(k')) \\ &= \sum_{n=n_{k'}}^{\infty} \left[\frac{1}{2} |\Delta y_n(k')|^2 + A(1 - \cos y_n(k')) \right] - \frac{1}{2} (\varepsilon_{k'})^2 - A(1 - \cos \varepsilon_{k'}). \end{aligned}$$

If there exists a $n_0 > n_{k'}$ such that $y_{n_0} \notin B_\gamma(\Theta)$, then $J(y(k')) - J(x(k')) \geq 3A/2 - (\varepsilon_{k'})^2/2 - A(1 - \cos \varepsilon_{k'})$. Otherwise, there exist two adjacent points such that the distance of them is larger than γ .

Then we have $J(y(k')) - J(x(k')) \geq \sum_{n=n_k}^{\infty} |\Delta y_n(k')|^2/2 - (\varepsilon_{k'})^2/2 - A(1 - \cos \varepsilon_{k'}) > 2\pi^2/9 - (\varepsilon_{k'})^2/2 -$

$A(1 - \cos \varepsilon_{k'})$. Define $\alpha := \min\{3A/2 - (\varepsilon_{k'})^2/2 - A(1 - \cos \varepsilon_{k'}), 2\pi^2/9 - (\varepsilon_{k'})^2/2 - A(1 - \cos \varepsilon_{k'})\} > 0$. We have $c_{\varepsilon_{k'}} = J(y(k')) \geq J(x(k')) + \alpha \geq c_{\varepsilon_{k'}} + \alpha$. This is a contradiction.

If Case 2 occurs, define

$$z_n(k) := \begin{cases} y_n(k), & n \leq m_k \\ \zeta, & n \geq m_k + 1 \end{cases}$$

Then $z(k) \in \Gamma_{\varepsilon_j}(\zeta)$ and

$$\begin{aligned} J(y(k)) - J(z(k)) &= \sum_{n=m_k}^{\infty} \left[\frac{1}{2} |\Delta y_n(k)|^2 + A(1 - \cos y_n(k)) \right] - \frac{1}{2} |\Delta z_{m_k}(k)|^2 - A(1 - \cos z_{m_k}(k)) \\ &= \sum_{n=m_k}^{\infty} \left[\frac{1}{2} |\Delta y_n(k)|^2 + A(1 - \cos y_n(k)) \right] - \frac{1}{2} \varepsilon_k^2 - A(1 - \cos \varepsilon_k) \end{aligned}$$

By applying a similar argument as in the Case 1, we get again a contradiction. \square

As we can see on Figure ??, every heteroclinic solution join two adjacent points of the set $\{2k\pi + \pi : k \in \mathbb{Z}\}$, or, after translation, heteroclinic solution join two adjacent points of the set $\{2k\pi : k \in \mathbb{Z}\}$. Denote by Υ the set of $\zeta \in \Theta$ such that there exist a heteroclinic solution joining 0 to ζ . The above observing gives $\Upsilon = \{-2\pi, 2\pi\}$, which will be proved strictly below. Since $A(1 - \cos x)$ is 2π -periodic, we have $J(y + 2\pi) = J(y)$. This implies that, for any integer $k > 0$, if there exists a heteroclinic orbit joining $-2k\pi$ and 0, there must exist a heteroclinic orbit joining 0 and $2k\pi$. Thus we need only to consider heteroclinic orbits joining 0 to $2k\pi$.

Lemma 2.6. $\Upsilon = \{-2\pi, 2\pi\}$.

Proof. Following the above argument, we just consider heteroclinic solutions joining 0 and $2k\pi$, where k is a positive integer. Suppose, to the contrary, Theorem 2.1 implies that there exist $\zeta = 2k\pi \in \Upsilon$ where $k > 1$. Lemma 2.3 guarantees existence of y which minimizes $J|_{\Gamma_{\varepsilon}(\zeta)}$. Denote $n_1 := \min\{m : y_n \in B_{\varepsilon}(\zeta), \forall n \geq m\}$. We have the following two cases:

Case 1: there exists n_0 such that $y_{n_0} = y_{n_1-1} - 2(k-1)\pi$.

Define

$$x_n = \begin{cases} y_n, & n \leq n_0 - 1 \\ y_{n+(n_1-n_0)} - 2(k-1)\pi, & n \geq n_0 \end{cases}$$

Then $x \in \Gamma_{\varepsilon}(2\pi)$ and

$$J(y) - J(x) = \sum_{n=n_0}^{n_1-2} \left[\frac{1}{2} |\Delta y_n|^2 + A(1 - \cos y_n) \right]$$

If $n_1 - 2 = n_0$, $J(y) - J(x) \geq 4(k-1)^2\pi^2$. Otherwise, there exists at least a suffix $n \in Z[n_0, n_1-2]$. If $n' \in Z[n_0, n_1-2]$ such that $y_{n'} \notin B_{\gamma}(\Theta)$, then we have $J(y) - J(x) \geq \frac{3}{2}A$. Otherwise, there must be two adjacent points such that the distant larger than γ . And then $J(y) - J(x) > \frac{2\pi^2}{9}$. Define $\beta := \min\{4(k-1)^2\pi^2, 3A/2, 2\pi^2/9\}$. All those situations contrary with $c_{\varepsilon}(\zeta) = c_{\varepsilon} \geq c_{\varepsilon}(2\pi) + \beta \geq c_{\varepsilon} + \beta$.

Case 2: If there is no n_0 such that $y_{n_0} = y_{n_1-1} - 2(k-1)\pi$, denote $n_2 := \max\{n : y_n < y_{n_1-1} - 2(k-1)\pi\}$ and two situation maybe meet:

Subcase I: If $n_2, n_1 - 1$ are two adjacent suffix.

Define

$$x_n = \begin{cases} y_n & n \leq n_2 \\ y_n - 2(k-1)\pi & n \geq n_2 + 1 \end{cases}$$

Then $x \in \Gamma_\varepsilon(2\pi)$ and

$$J(y) - J(x) = \frac{1}{2}|\Delta y_{n_2}|^2 - \frac{1}{2}|\Delta x_{n_2}|^2 > 2(k-1)^2\pi^2,$$

which implies the following contradiction

$$c_\varepsilon(\zeta) = c_\varepsilon \geq c_\varepsilon(2\pi) + \beta \geq c_\varepsilon + \beta.$$

Subcase II: $n_2 < n_1 - 2$. Then, we have $y_{n_2} < y_{n_1-1} - 2(k-1)\pi < y_{n_2+1}$.

Define

$$x_n = \begin{cases} y_n & n \leq n_2 \\ y_{n+(n_1-n_0)} - 2(k-1)\pi & n \geq n_2 + 1 \end{cases}$$

Then $x \in \Gamma_\varepsilon(2\pi)$ and

$$\begin{aligned} J(y) - J(x) &= \sum_{n=n_2}^{n_1-2} \left[\frac{1}{2}|\Delta n|^2 + A(1 - \cos y_n) \right] - \frac{1}{2}|\Delta x_{n_2}|^2 - A(1 - \cos x_{n_2}) \\ &= \sum_{n=n_2+1}^{n_1-2} \left[\frac{1}{2}|\Delta y_n|^2 + A(1 - \cos y_n) \right] + \frac{1}{2}|\Delta y_{n_2}|^2 - \frac{1}{2}|y_{n_1-1} - 2(k-1)\pi - y_{n_1}|^2 \end{aligned}$$

A similar argument as Case 1 of Theorem 1 induces a contradiction.

Consequently, we finish our proof. \square

Theorem 2.2. *For each $\xi \in \Theta$, there exist at least two heteroclinic orbits joining $\xi - 2\pi$ to ξ and at least two of heteroclinic orbits joining ξ to $\xi + 2\pi$.*

Proof. Without loss generality, we only need to check heteroclinic orbits joining 0 and $\zeta \in \Upsilon$. Lemma 2.6 implies that only -2π and 2π belong to Υ . If $\{y_n\}$ is a heteroclinic orbit connecting 0 and 2π , then $\{y_{-n}\}$ is also a heteroclinic solution joining 2π to 0. And $\{y_n - 2\pi\}, \{y_{-n} - 2\pi\}$ also two heteroclinic solutions joining -2π to 0. The proof is complete. \square

3 Reasons for choosing Phase Plane of (1) with $A = 0.1$

For simplicity, we paint phase plane of (2) with $A = 1$ in section 1. We should paint phase plane of (1) with $A = 1$ to compare with that of (2). However, phase plane of (1) with $A = 1$ (figure ??) is so different from that of (2). Non-periodic solutions move between the upper and lower half plane of (1). At first glance, the phase plane of (1) is different from that of (2) in essence. But it is not. Those phenomena appear because of approximation error. Approximation error depends on amplitude. When amplitude A equal to 10, we paint the phase plane of (1) as figure ??. All periodic solutions and non-periodic solutions become disordered. That is why we choose the phase plane of (1) with $A = 0.1$.

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